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EXTREMAL SET SYSTEMS WITH WEAKLY RESTRICTED INTERSECTIONS

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Problems concerning extremal set systems with intersections of restricted cardinality are probably among the most popular problems in extremal combinatorics, leading to many surprising discoveries and exciting questions. In this paper, we discuss the "weak" versions of some problems of this type, where the restricted intersection property is weakened by the possible existence of some (or maybe many) intersections having "exceptional" sizes. In particular, we prove a tight upper bound for a weak version of the "odd town" problem. We also give a tight bound for a weak version of the nonuniform Fisher inequality and see how the proof of this bound leads to an extremal set theoretic characterization of Hadamard's matrices. Finally, we display a tight bound for a weak version of the "even town" problem, and use this bound to tackle problems concerning systems with restricted multi-intersections.

1. Introduction

Let X be a set of n elements and L be a (not necessarily finite) set of non-negative integers. A family $\mathcal{F}$ of subsets of X is called a set system with restricted intersections (with respect to L) if the cardinality of the intersection of any two members in $\mathcal{F}$ belongs to L. Given n and L, we denote by $m(n,L)$ the maximum size of such a family. The problem of estimating $m(n,L)$ for various L results in many popular and powerful theorems. In the following, let us describe some of these.

The first and probably best known result in this area is the classical nonuniform Fisher inequality [3,10], which asserts that if all intersections have the same cardinality, then the system cannot have more than n members. In our terminology, it says

Theorem 1.1. *If L contains only one number, then*

$$m(n,L) \le n.$$

The following theorem, proven independently by Berlekamp [4] and Graver [9], determines $m(n,L)$ in the case L is the set of all non-negative even integers. This theorem is occasionally referred to as the "even town" theorem [3].

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Theorem 1.2. *If L is the set of all non-negative even integers, then*

$$m(n, L) = 2^{\lfloor n/2 \rfloor} + \delta(n),$$

where $\delta(n) = 0$ if n is even and $\delta(n) = 1$ if n is odd.

Quite often, one may require additional conditions on the cardinality of the members in the family. The famous Erdős–Ko-Rado theorem and the following theorem ([3]) belong to this category.

Theorem 1.3. *Assume that all members in a family $\mathcal{F}$ have odd cardinality and every two members have even intersection, then $\mathcal{F}$ cannot have more then n members.*

As a match to Theorem 1.2, Theorem 1.3 is usually referred to as the "odd town " theorem [3]. There are many other wonderful theorems in this area such as the Erdős–de Bruijn inequality [6], the Ray-Chaudhury–Wilson theorem [13], the Frankl–Wilson theorem [8], etc. We will discuss some of these results in later sections.

The most amazing about theorems on set systems with restricted intersections is the fact that many of them can be proven in a very elegant way by the linear algebraic method, using the following general scheme:

Step 1. Assign vectors of a properly chosen linear space V to the members of the system.

Step 2. Use the restricted intersection condition to show that the family of vectors should have certain properties, and then use these properties to derive a bound for the size of the family. For instance, the most common trick is to show that the vectors in the family are independent, which implies that the cardinality of the family cannot exceed the dimension of the space.

Step 3. If the bound is not yet tight, one can occasionally sharpen it by adding appropriate vectors to the system and applying Step 2 to the new (extended) set of vectors. By this trick one gains an additional term (which is equal to the number of vectors added) on the bound.

The excellent monograph [3], by Babai and Frankl, contains many proofs using this scheme. Readers could also find here various astonishing applications of theorems on systems with restricted intersections in many areas of mathematics.

Let us sketch here the proof of Theorem 1.3, which illustrates the ideas described in the scheme very nicely.

Proof of Theorem 1.3. To each member A of the set system, consider its characteristic vector v_A in $GF^n(2)$, the n-dimensional vector space over the field $GF(2)$. The even intersection condition implies that if $A \neq B$, then $v_A v_B = 0$, where $v_A v_B$ denotes the classical inner product of v_A and v_B.

On the other hand, every member in the system has odd size, thus $v_A v_A = 1$ for every A. Now we prove that the vectors v_A are independent. Indeed, assume that $v_{A_1} + \ldots + v_{A_k} = 0$. Taking the inner product of both sides with v_{A_1}, we would obtain

$$v_{A_1} v_{A_1} + v_{A_1} v_{A_2} + \ldots + v_{A_1} v_{A_k} =$$

$$= 1 + 0 + \ldots + 0 = 1 = v_{A_1} 0 = 0,$$

a contradiction. Thus, the vectors v_A are independent; this implies that the size of the system is at most n, the dimension of the space, and completes the proof. ∎

Looking at this proof carefully, we can see that Step 2 is crucial, and at this step we definitely need to use the restricted intersection property. One may ask the following natural question:

Can we still prove some bounds if we allow some exceptional intersections of sizes not in the restricted set L?

In general, it is not clear how one could deal with this relaxation. Because of the existence of exceptional intersections, the statements we can prove (in Step 2) under the original condition will not hold. For instance, if odd intersections exist, then the argument used in the proof of Theorem 1.3 could not be repeated by any chance.

Fortunately, the situation is not completely hopeless. In many cases, it turns out that if the number of exceptional intersections is relatively small, then the problem is still solvable with the help of additional tools (from extremal graph theory, for instance). In these cases we are able to prove tight bounds, which, in addition, lead to interesting optimal constructions. The purpose of this paper is to present these possibilities and to point out further questions that may lead to a deeper understanding of the subject.

First, we need to state the problem precisely. Let s be a non-negative integer and L be a set of non-negative integers; we want to determine $m_s(n,L)$, the maximum cardinality of a set system $\{A_1, A_2, \ldots, A_m\}$ on the ground set X of n elements with the property that for each index i there are at most s indices $j \neq i$ satisfying $|A_i \cap A_j| \notin L$. In other words, each A_i could have at most s "exceptional" intersections. For the special case when $s = 0$, we have $m_0(n,L) = m(n,L)$, and we return to the original problem when every intersection is restricted. If $s > 0$ we call the problem the s-weak version of the original. If there are further conditions on the sizes of the members of the family (such as in Theorem 1.3), we shall keep the general notation $m_s(n,L)$ unchanged, and specify these conditions separately. The rest of the paper is organized as follows.

In Section 2, we consider the s-weak version of Theorem 1.3. We give a tight bound for the case $s < 2^{n/4}$ and an asymptotically tight bound for $s \gg 2^{n/2}$. The proof involves a tricky combination of various tools from linear algebra and extremal graph theory.

Section 3 starts with a weak version of Theorem 1.2. In subsection 3.1, we display a tight bound for the case $s < c2^{n/2 - \log n}$, where c is a positive constant. This result is proven in a previous paper [14]. The rest of Section 3 is devoted to problems concerning systems with even multi-intersections (intersections of more than two sets) and their weak versions. We shall briefly discuss the difficulties one may have with intersections of more than two sets. For instance, we shall explain why the general proving scheme presented above, which is extremaly powerful in dealing with intersections of two sets, becomes useless for multi-intersections. Quite surprisingly, the notion of weakly restricted intersection turns out to be useful in the study of systems with restricted multi-intersections, at least in some special cases.

In Section 4, we consider the 1-weak version of the nonuniform Fisher inequality (Theorem 1.1). Although the Fisher inequality is probably the best known theorem in the area, the classification of the optimal systems (those that have exactly n members) is still far from completed. The main conjecture, known as the λ-design conjecture (by Ryser and Woodall [12, 15]), is almost thirty years old, but still open. Rather unexpected, for the 1-weak version, we could find both the tight upper bound and the classification of all optimal systems. This classification leads us to a new extremal set theoretic characterization of Hadamard's matrices.

In Section 5, we present a few more results concerning weak versions of other classical problems along with several open questions and conjectures.

2. Weak odd town

O dd town is a village with n lazy inhabitants, spending their time forming clubs. These clubs should obey the following rules:

1. Each club should have an odd number of members.
2. Every pair of clubs should have an even number of members in common.

Theorem 1.3 asserts that this crazy village cannot create more than n clubs. We now wonder how many more clubs could they create, if rule 2 is weakened in the way we described in the introduction. Assume that each club may have at most s odd intersections with other clubs. We are interested in $m_s(n,L)$, the maximal number of clubs the village may have under the modified rules.

In this and the following section log has base 2 and $\{a\}$ denotes the fraction part of a, that is, $\{a\}=a-\lfloor a\rfloor$. We shall sometimes call a set odd (even) if it has odd (even) cardinality. Our main result in this section is the following.

Theorem 2.1. *If s satisfies the following two conditions*

- $\lceil \log s\rceil \le (n/4)-1$
- *If* $\{\log s\}\neq 0$ *then* $\{\log s\}\ge \log(1+(8/n))$,

then $m_{s-1}(n,L)=s(n-2\lceil \log s\rceil)$.

The first condition on s indicates that the theorem may not hold for very large s. This turns out to be the case, as shown in Theorem 2.3. However, the rather weird looking second condition may need some explanation, and we shall address this issue in a remark at the end of the proof. To make the appearance of Theorem 2.1 cleaner, we consider $m_{s-1}(n,L)$ instead of $m_s(n,L)$.

Proof. Let $\{A_1,\dots,A_m\}$ be a set system satisfying the modified rules. Let v_i denote the characteristic vector of A_i in $GF^n(2)$. Define a simple graph $G(V,E)$ on the set $V=\{1,2,\dots,m\}$ as follows: $(i,j)\in E$ if and only if $|A_i\cap A_j|\equiv 1 \pmod 2$, i.e., $v_iv_j=1$.

Since we are considering $m_{s-1}(n,L)$, each A_i has at most s–1 odd intersections. Consequently, $\deg i < s$ for every i.

Consider an independent set D in G with maximum cardinality $|D| = \alpha$. By a well-known theorem of Turán, it follows that $\alpha \geq m/s$, so $r = \alpha - m/s \geq 0$. For each $i \in D$ let T_i be the set consisting of i and all the vertices of G the only neighbor of which in D is i. Let $T = V \backslash \cup_{i\in D} T_i$. Denote by e the number of edges between the two components D and $V \backslash D$. Since $\deg i < s$ for every $i \in D$, we have $\alpha(s-1) \geq e$.

To bound e from below, notice that (by the definition of D) any point in $V \backslash D$ should have a neighbor in D. Moreover, if a point is in T, then it should have at least two neighbors in D. Therefore, $e \geq |T| + (m-\alpha)$. Consequently, $\alpha(s-1) \geq |T| + (m-\alpha)$. Replacing $\alpha = m/s + r$, we obtain $rs \geq |T|$.

Notice that $\sum_{i\in D} |T_i| = m - |T|$. Without loss of generality, one can suppose that $1 \in D$ and $|T_1| = \max_{i\in D} |T_i|$. This implies that $|T_1| \geq (m - rs)/\alpha$.

Consider the subgraph G_1 induced by T_1. If there were a pair $j, j' \in T_1$ such that $(j,j') \notin E$, then $D \cup \{j,j'\} \backslash \{1\}$ would be an independent set of cardinality larger than α, which contradicts the definition of D. This shows that G_1 should be the complete graph. Let j, j', k, k' be arbitrary elements of T_1, the definitions of G and D imply that

$$(v_j + v_{j'})(v_k + v_{k'}) = v_j v_k + v_j v_{k'} + v_{j'} v_k + v_{j'} v_{k'} = 1+1+1+1 = 0$$
$$(v_j + v_{j'}) v_1 = v_j v_1 + v_{j'} v_1 = 1+1 = 0$$
$$(v_j + v_{j'}) v_i = v_j v_i + v_{j'} v_i = 0+0 = 0$$

for every $i \in D \backslash \{1\}$. (The equalities are in $GF(2)$.)

Denote by V_1 the subspace of $GF^n(2)$ spanned by the vectors $v_j + v_{j'}$, where $j, j' \in T_1$. Let $\mathcal{D} = \{v_i | i \in D\}$. The equalities above imply that $V_1 \subset V_1^{\perp}$ and $\mathcal{D} \subset V_1^{\perp}$.

Here comes the key trick of the proof. By a well-known fact from linear algebra, there is a subspace $V_2 \subset V_1^{\perp}$ such that $V_1^{\perp} = V_1 \oplus V_2$, where $\oplus$ denotes the direct sum. It follows that each vector $v_i \in \mathcal{D}$ has a unique decomposition $v_i = v_i^1 + v_i^2$ with $v_i^1 \in V_1$ and $v_i^2 \in V_2$.

Note that for each pair $i, j \in D$, $v_i v_j = \delta_{ij}$, where δ_{ij} is the Kronecker index; $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise. On the other hand, by the decomposition and the fact that $V_1 \subset V_1^{\perp}$

$$v_i v_j = (v_i^1 + v_i^2)(v_j^1 + v_j^2) = v_i^2 v_j^2,$$

which implies that $v_i^2 v_j^2 = \delta_{ij}$. Following the proof of Theorem 1.3, one can use this fact to show that the vectors v_i^2, where $v_i \in \mathcal{D}$, are independent. So

$$|\mathcal{D}| = \alpha = \frac{m}{s} + r \leq \dim\, V_2 = \dim V_1^{\perp} - \dim V_1 = n - 2\dim\, V_1,$$

which implies $m \le s(n-r-2\dim V_1)$. Since V_1 contains at least $|T_1|$ different vectors, $\dim V_1 \ge \lceil \log|T_1| \rceil \ge \lceil \log((m-rs)/\alpha)\rceil$. Let $f(r) = r + 2\lceil \log \frac{(m-rs)}{\alpha} \rceil$, where $\alpha = (m/s)+r$. With this notation the fact that $m \le s(n-2\dim V_1)$ implies that $m \le s(n-f(r))$.

We complete the proof indirectly. Assume (for a contradiction) that $m > s(n-2\lceil \log s\rceil)$; since $m \le s(n-f(r))$, it follows that $f(r) < 2\lceil \log s \rceil$. We shall show that this will actually lead to a contradiction. Consider the following two cases:

Case 1. $r > (m/s)-4$. By the first condition on s that $\lceil \log s \rceil \le (n/4)-1$, it follows that $2\lceil \log s\rceil \le (n-4)/2$ and hence

$$s(n-2\lceil \log s\rceil) \ge \frac{s(n+4)}{2} \ge \frac{s(\alpha+4)}{2} = \frac{s(\frac{m}{s}+r+4)}{2} \ge \frac{s(\frac{m}{s}+\frac{m}{s}-4+4)}{2} = m,$$

a contradiction.

Case 2. $r \le (m/s)-4$. Suppose $r \ge 2$, note that then $(\frac{m}{s}-r)/(\frac{m}{s}-r+2) \ge 4/6$ and $(\frac{m}{s}+r-2)/(\frac{m}{s}+r) \ge 6/8$. Moreover

$$\log \frac{m-rs}{(m/s)+r} - \log \frac{m-(r-2)s}{(m/s)+(r-2)}$$
$$= \log \frac{m-rs}{(m/s)+r} \times \frac{(m/s)+(r-2)}{m-(r-2)s} \ge \log \frac{4\times 6}{6 \times 8} = -1$$

which implies that $f(r) \ge f(r-2)$. Let $r' = r - 2\lfloor r/2 \rfloor$. We have $0 \le r' < 2$ and $f(r') \le f(r) < 2\lceil \log s\rceil$. The fact that $r' < 2$ yields the following

$$f(r') \ge 2\lceil \log s \frac{(m/s)-r'}{(m/s)+r'} \rceil \ge 2\lceil \log s \frac{n-2\lceil \log s\rceil - r'}{n-2\lceil \log s \rceil + r} \rceil$$
$$\ge 2\lceil \log s \frac{n-2((n/4)-1)-2}{n-2((n/4)-1)+2} \rceil = 2\lceil \log \frac{s}{1+(8/n)} \rceil.$$

Now assume that $s = 2^{k+\delta}$, where $0 \le \delta < 1$ and k is an integer. The second condition on s yields

$$\frac{s}{1+(8/n)} = 2^k \frac{2^\delta}{1+(8/n)} > 2^k,$$

which shows that $f(r') \ge 2(k+1) = 2\lceil \log s \rceil$, a contradiction. This shows that the assumption $m > s(n-2\lceil \log s \rceil)$ is false; thus, we can conclude that $m_{s-1}(n,L) \le s(n-2\lceil \log s\rceil)$.

To show the lower bound, let us consider the following construction.

Construction 2.2. Consider the set $X = \{a_1, a_2, \ldots, a_p, b_1, b_2, \ldots, b_p, c_1, c_2, \ldots, c_l\}$, where $p = \lceil \log s \rceil$ and $l = n-2p$. It is trivial that X has n elements. Choose s

different subsets $I_1, I_2, \ldots, I_s$ of the set $\{1,2,\ldots,p\}$ and set $A_{i,j} = \{c_i\} \cup \bigcup_{q \in I_j} \{a_q, b_q\}$ for every $1 \le i \le l, 1 \le j \le s$. This way we obtain a system of $s(n-2p)$ sets satisfying the desired weak restricted intersection property. Indeed, it is trivial that $|A_{i,j}|$ is odd for all i,j. Moreover, $|A_{i,j} \cap A_{i',j'}|$ is odd if and only if $i = i'$. Since j' can have $s-1$ values different from j, it follows that each member of the system has exactly $s-1$ odd intersections. This shows that the bound is tight and completes our proof. ∎

Remark. In Theorem 2.1, we want to find the exact value of $m_{s-1}(n,L)$, which is an integer. This requires the round-off of $\log s$ and results in some technical arguments at the end of the proof. A slightly weaker bound $m_{s-1}(n,L) \le s(n-2\log s)$ could be proven with less technical details and without the (somewhat artificial) condition that $\{\log s\} \ge \log(1+(8/n))$. This condition was introduced in order to keep s away from a power of 2 so that the function $s(n-2\lceil \log s \rceil)$ is increasing in the restricted domain of s. This is clearly a necessary condition, since (by definition) $m_s(n,L)$ is an increasing function in s. It is easy to see that if s is very close to a power of 2, then $s(n-\lceil \log s \rceil)$ is not increasing. For example, $s(n-2\lceil \log s \rceil) < (s-1)(n-2\lceil \log(s-1) \rceil)$, for any $s = 2^k+1 > n$. The following theorem deals with the case when s takes larger values.

Theorem 2.3. 1. *If* $2^{n-2} > s > 2^{n/2}$ *then* $m_s(n,L) = 2s + O(2^{n/2})$.

2. *If* $2^{n/2} \ge s \ge 2^{n/4}$ *then* $m_{s-1}(n,L) < s(n/2+1.5)$.

The first bound is approximately tight in the sense that when $s \gg 2^{n/2}$, the error term is negligible. For $s \ge 2^{n-2}$, we have $m_s(n,L) = 2^{n-1}$ because in this case the family of all 2^{n-1} subsets of odd cardinality satisfies the weak intersection property.

Proof. To prove the first part of the theorem, let us consider a graph G defined on the family of 2^{n-1} subsets of odd cardinality of a ground set X ($|X| = n$, as usual) as follows: Two odd subsets form an edge if their intersection is also odd. One could prove that G has a pseudo-random property that for every subgraph G' on n' nodes, G' has $\binom{n'}{2}/2 + O(2^{n/2}n')$ edges. This means that if a subgraph has more than $2^{n/2}$ vertices then its average degree is about half of the number of vertices. Consequently, if the maximum degree of a subgraph is at most s, then the number of vertices could exceed $2s$ by only a term of order $O(2^{n/2})$. This implies

$$m_{s-1}(n,L) \le 2s + O(2^{n/2}).$$

For general information about pseudo-random graphs, we refer to [2]. It is interesting to compare the first part of Theorem 2.3 to Theorem 2.1, and observe the fundamental shift in the background of the problem. For smaller s, the problem has algebraic flavor; but when s getting larger it starts to merge into the area of random structures.

To show that the bound is sharp up to the error term $O(2^{n/2})$, let us consider the following construction.

Constructions 2.4. Suppose $s=2^{p_1}+2^{p_2}+\ldots+2^{p_k}$, where $n-2>p_1>p_2>\ldots>p_k\geq 0$. Consider a family of nested sets $A_1\supset A_2\supset\ldots\supset A_k$, where $|A_i|=p_i+3$ for $i<k$ and $|A_k|=p_k+2$. For all $1\leq i<k$, choose $x_i\in A_i\backslash A_{i+1}$, and let $\mathcal{A}_i$ be the family of all subsets of odd cardinality of A_i containing x_i. Furthermore, let $\mathcal{A}_k$ be the family of all odd subsets of A_k. By the way x_i were chosen, it is obvious that the families $\mathcal{A}_i$ are mutually disjoint. It is also simple to verify that $|\mathcal{A}_i|=2^{p_i+1}$ for all $1\leq i\leq k$. The family $\mathcal{A}=\cup_{i=1}^k\mathcal{A}_i$ has exactly $\sum_{i=1}^k 2^{p_i+1}=2s$ members, and readers could check without much difficulty that each member in this family has at most s odd intersections.

To prove the second part of the theorem, let us repeat the proof of Theorem 2.1 with the same notations. This way one can deduce that $m_{s-1}(n,L)\leq s(n-h(r))$, where $h(r)=r+2\log\frac{(m-rs)}{\alpha}$ and $\alpha=(m/s)+r$. It is easy to verify that $h'(r)=1-\frac{4m/s}{(m/s+r)(m/s-r)}$.

Note that if $r<(m/s)-3$ then $h'(r)>0$, i.e., $h(r)$ is increasing. Hence, $h(r)\geq h(0)=2\log s$. It follows that $m_{s-1}(n,L)\leq s(n-h(r))\leq s(n-h(0))=s(n-2\log s)\leq sn/2$.

In the remaining case when $r\geq(m/s)-3$, notice that by definition $r=\alpha-(m/s)$. Thus $r\geq(m/s)-3$ implies $\alpha\geq 2(m/s)-3$. On the other hand, by the definition of α ($\alpha=|D|$ = the maximum size of an independent set in G) one can show that $n\geq\alpha$, using the argument that the characteristic vectors of the members in D are independent. Hence, we have $n\geq 2(m/s)-3$, that is, $m\leq s(n/2+1.5)$. The proof is thus completed. ∎

3. Set systems with even multi-intersections

3.1. Weak even town

Erdős asked the following question [4]: *How many subsets can one choose from a set of n elements so that the intersection of every two subsets is even?*

Theorem 1.2, proven independently by Berlekamp [4] and Graver [9], gives the complete answer for this question. In our terminology, it states that if L is the set of all non-negative even integers, then $m_0(n,L)=2^{\lfloor n/2\rfloor}+\delta(n)$, where $\delta(n)=1$ if n is odd and $\delta(n)=0$ otherwise.

In [14], we solved the weak version of this problem for all values of s up to $c2^{n/2-\log n}$ with some positive constant c. The following theorem is a simplified and more pleasant version of the result stated in [14].

Theorem 3.1.2. *There is a positive constant c so that*

1. *If n is odd and $s < c2^{n/2-\log n}$ then $m_s(n,L) = 2^{\lfloor n/2 \rfloor} + s$,*
2. *If n is even and $s < c2^{n/2-\log n}$ then $m_s(n,L) = 2^{\lfloor n/2 \rfloor}$.*

In the next section, we shall apply Theorem 3.1.2 to tackle a problem concerning systems with even multi-intersections and its weak versions.

3.2. Systems with even multi-intersections

The following problem can be considered as a generalization of Erdős's problem mentioned above.

Let k be a fixed integer at least 2. How many subsets can one choose from a set of n elements such that the intersection of any k of them is even?

As mentioned in the introduction, problems concerning intersections of more than 2 sets are definitely harder than those concerning intersections of two sets. In fact, very few theorems on systems with restricted multi-intersections are known, and no general proving scheme such as the scheme described in the introduction exists. In the next few lines, let us briefly address the main obstruction caused by multi-intersections. The power of the proving scheme described in Section 1 relies on the fact that usually one can find a *binary* operator H from $V \times V$ to a properly chosen field, so that the value of $H(u,v)$ carries some significant information about the intersection of the sets represented by u and v. For instance, in the proof of Theorem 1.3, $H(u,v)$ is the inner product of u and v in $GF(2)$, and its value gives us full information about the parity of the intersection of the corresponding sets. However, for the intersection of three sets, we have not yet found any natural *ternary* operator to carry out the same job. And the problem does not look any simpler when one wants to consider intersections of more sets.

Quite surprisingly, it turns out that in our current problem the notion of set systems with weakly restricted intersections provides a great help. Using this notion, we could determine exactly the maximal cardinality of the set system in question, when k is not too large (compared to n). The following theorem was proven in [14].

Theorem 3.2.1. *There is a positive constant c so that the following holds. If $A_1, \dots A_m$ are different subsets of a set X of $n > c \log k$ elements with the property that for every set of k different indices $i_1, \dots, i_k$, the intersection $\cap_{j=1}^{k} A_{i_j}$ has even cardinality, then $m \le 2^{\lfloor n/2 \rfloor}$ in the case n is even and $m \le 2^{\lfloor n/2 \rfloor} + k - 1$ in the case n is odd.*

The proof of this theorem relies on the following lemma.

Lemma 3.2.2. *If $A_1, A_2, \dots, A_m$ are odd subsets of a set X of n elements with the property that for every k different indices $i_1, \dots, i_k$, the intersection $A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_k}$ has even cardinality, then $m \le (k-1)(n - \log(k-1))$.*

To prove Theorem 3.2.1, we first bound the number of odd intersections of any member in the family using the previous lemma, then apply Theorem 3.1.2, which is a theorem on systems with weakly restricted intersections. For the detailed proof, which involves many technical arguments in the case of odd n, we refer to [14].

Let us now consider the s-weak version of Theorem 3.2.1. The problem reads as follows.

Let $A_1,\dots,A_m$ be subsets of a set of n elements such that for every set of $k-1$ different indices $i_1,i_2,\dots,i_{k-1}$, there are at most s indices i_k ($i_k \neq i_j, j<k$) such that $|\cap_{j=1}^{k} A_{i_j}|$ is odd. What is the maximum value of m in terms of s, k and n?

Again, we are able to prove a sharp bound, assuming s and k are not too large. The following theorem generalizes Theorem 3.2.1.

Theorem 3.2.3. *There is a positive constant c so that the following holds. If $A_1,\dots,A_m$ are different subsets of a ground set X of $n > c(\log(s+k))$ elements with the property that for every set of $k-1$ different indices $i_1,i_2,\dots,i_{k-1}$, there are at most s indices i_k ($i_k \neq i_j, j<k$) such that $|\cap_{j=1}^{k} A_{i_j}|$ is odd, then $m \leq 2^{\lfloor n/2 \rfloor}$ in the case n is even and $m \leq 2^{\lfloor n/2 \rfloor}+k+s-1$ in the case n is odd.*

In order to prove this theorem, we first need to prove the following lemma, which can be seen as the s-weak version of Lemma 3.2.2.

Lemma 3.2.4. *If $A_1,A_2,\dots A_m$ are odd subsets of a ground set of n elements with the property that for every $k-1$ different indices $i_1,i_2,\dots,i_{k-1}$, there are at most s indices i_k ($i_k \neq i_j, j<k$) such that $|\cap_{j=1}^{k} A_{i_j}|$ is odd, then $m<(k-1+s)(n-\log(k-1))$.*

Proof. If $k>m$, then the statement is trivial. Therefore, we may assume that $k \leq m$. For each A_i, let v_i be its characteristic vector in $GF^n(2)$; furthermore, let d be the dimension of the subspace spanned by the vectors v_i. Clearly, $d \leq n$. Moreover, notice that in any subspace of $GF^n(2)$ at least half of the vectors have even weights. Since all v_i have odd weights, we conclude that $2^{d-1} \geq m \geq k$. Thus, $d \geq \log k+1 > \lfloor \log(k-1) \rfloor +1$.

For any $d > \lfloor \log(k-1) \rfloor + 1 = d_0$ and any $k \geq 2$, we shall prove a stronger inequality that

(3.2.4.1). $$m \leq (k-1+s)(d-\log(k-1))$$

In the proof of (3.2.4.1), we shall need the following elementary claims.

Claim 1. *If $d=d_0+1$ then*

$$(k-1)(d-\log(k-1)) \geq 2^{d-1}.$$

Proof. Let $x=\lfloor \log(k-1) \rfloor +1-\log(k-1)$. It is clear that $0<x\leq 1$. The claim is equivalent to the following

$$(k-1)2^x \leq (k-1)(1+x),$$

which follows immediately from the easy fact that for $0<x\leq 1$, $2^x \leq 1+x$.

Claim 2. *If* $d \geq \log(k-1)+2$ *then*

$$(k-2)(d-\log(k-2)) \leq (k-1)(d-\log(k-1)).$$

Proof. The claim is equivalent to

$$(k-2)(\log(k-1)-\log(k-2)) \leq d-\log(k-1).$$

Since the function $x\log(1+1/x)$ is increasing, it is easy to see that the left hand side is upper bounded by $\log e < 2$ (where e is the natural logarithmic base). Since $d-\log(k-1) \geq 2$, the proof is complete.

The proof of (3.2.4.1) uses induction on both k and d. To start, let us notice that if $d=d_0+1$, then by Claim 1.

$$m \leq 2^{d-1} \leq (k-1)(d-\log(k-1)) \leq (k-1+s)(d-\log(k-1)),$$

for any $k \geq 2$ and $s \geq 0$. This says that (3.2.4.1) holds in this case for any $k \geq 2$. Now let us consider the case $k=2$. Define a graph G on $\{1,2,\dots,m\}$ the following way. For any $1 \leq i < j \leq m$, (i,j) forms an edge if and only if $A_i \cap A_j$ is odd, or in other words, $v_i v_j = 1$. By the weak restricted intersection of the system, each vertex has degree at most s. Thus, there is an independent set I of size at least $m/(s+1)$. Without loss of generality, assume $I=\{1,2,\dots,t\}$, where $t \geq m/(s+1)$. By the definition of G, the system $\{A_1,\dots,A_t\}$ satisfies the condition of Theorem 1.3. Therefore, we can repeat the proof of Theorem 1.3. to show that $t \leq d$. This implies that

$$m/(s+1) \leq t \leq d,$$

which yields $m \leq (s+1)d = (k-1+s)(d-\log(k-1))$, for $k=2$. Hence, we may conclude that (3.2.4.1) holds if $k=2$ for any legitimate value of d.

Consider $k>2$ and $d>d_0+1$. Assume that (3.2.4.1) holds for any pair (k',d'), where (k',d') satisfies one of the following two conditions: (1) $d_0+1 \leq d' < d$ and $k' \geq 2$; (2) $2 \leq k' < k$ and $d' \geq d_0+1$. We shall prove that it holds for the pair (k,d). Consider the following two cases:

Case 1. For every set of $k-1$ different indices $i_1,\dots,i_{k-1}$, the sets $A_{i_1},\dots,A_{i_{k-1}}$ have even multi-intersection. In this case, for any set of $k-2$ different indices $i_1,\dots,i_{k-2}$, there is no i_{k-1} such that the intersection $\cap_{j=1}^{k-1} A_{i_j}$ is odd. Since $d > d_0+1$, it follows that $d \geq \log(k-1)+2$. Thus, by the induction hypothesis (applied on k) and Claim 2, we have

$$m \leq (k-2)(d-\log(k-2)) \leq (k-1)(d-\log(k-1)) \leq (k-1+s)(d-\log(k-1)).$$

Case 2. There are $k-1$ sets $A_{i_1},\dots,A_{i_{k-1}}$ with odd intersection. Let $A_{j_1},\dots,A_{j_u}$ be the members of the system which have odd intersection with $A=\cap_{j=1}^{k-1} A_{i_j}$. By the restricted intersection property of the system, $u \leq s$. Call a vector v_l *busy* if it is the characteristic vector of one of the $k-1+u$ members considered above. If v_l is not busy, we say that v_l is *free*.

Claim 3. *Any busy vector is independent from the subspace spanned by the free vectors.*

Proof. Let v denote the characteristic vector of A. Suppose v_l is free; by the definition of a free vector,

$$v_l v = |A_l \cap A| \pmod 2 = 0.$$

On the other hand, if v_l is busy, then $A_l \cap A$ has odd cardinality by the definition of busy vectors. Thus

$$v_l v = |A_l \cap A| \pmod 2 = 1.$$

Now one can complete the proof by a similar argument as in the proof of Theorem 1.3.

From Claim 3, it follows that the dimension of the subspace spanned by the free vectors is at most $d-1$. Thus, by the induction hypothesis (applied on d), the number of free vectors is at most $(k-1+s)(d-1-\log(k-1))$. On the other hand, the number of busy vectors is $k-1+u \leq k-1+s$. Together, we obtain

$$m \leq (k-1+s)(d-1-\log(k-1)) + (k-1+s) = (k-1+s)(d-\log(k-1)),$$

which completes the proof of (3.2.4.1) and that of the lemma. ∎

Now let us sketch the proof of Theorem 3.2.3 for the case n is even. Fix an arbitrary member A_i, and let $B_1, \dots, B_l$ be its odd intersections with other members of the system. One can verify that the family $\{B_j\}_{j=1}^{l}$ satisfies the following two conditions:

- $|B_j|$ is odd (by definition).
- For any $k-2$ sets $B_{i_1}, \dots, B_{i_{k-2}}$, there are at most s sets B_j $(j \neq i_p)$ so that $B_j \cap \bigcap_{r=1}^{k-2} B_{i_r}$ has odd cardinality.

By Lemma 3.2.4, we have the following bound on l:

$$l < (k-2+s)(n-\log(k-2)) < (k+s)n.$$

So, any member A_i of our system cannot have more than $(k+s)n$ odd intersections with other members. By Theorem 3.1.2, if the condition $(k+s)n < c2^{n/2}/n$ is satisfied, then our system cannot have more than $2^{n/2}$ members. This completes the proof of Theorem 3.2.3 for the case of even n.

To get the bound in the case of odd n, one needs to repeat the proof of Theorem 3.2.1 (see [14]) with some nominal modification. We omit the details.

To see that the bounds in Theorem 3.2.3 are sharp, consider the following construction. Assume that n is odd, partition a subset of n–1 elements into $(n$–$1)/2$ pairs. Let $\mathcal{B}$ be the family consisting of all possible unions of pairs ($|\mathcal{B}| = 2^{(n-1)/2}$). Choose $k+s-1$ arbitrary members of $\mathcal{B}$ and add to each of them the remaining element of the ground set. The union of $\mathcal{B}$ and these $k+s-1$ subsets forms a family achieving the upper bound. For n even, simply take all possible unions of $n/2$ pairs. This family has $2^{n/2}$ members and every k of them have even intersection, no matter what k is.

4. Non-uniform Fisher inequality under weak condition

Let $L=\{\lambda\}$, where λ is a non-negative integer. Theorem 1.1, known as the nonuniform Fisher inequality [5,10], states that $m_0(n,L)\le n$. Although this is probably the first and most famous result on set systems with restricted intersections, it is still difficult to characterize all optimal systems (when equality is achieved). There are characterizations for some special cases; the best known is probably the theorem of Erdős and de Bruijn which deals with the case $L=\{1\}$ [6]. Under this circumstance, an optimal system is either a projective plane or a "pencil". If the system is uniform (i.e., all members have the same cardinality), it is known that an optimal system should be a symmetric design [11]. The situation in the general case is unclear. In order to classify all optimal systems without special assumptions, Ryser and Woodall (independently) made the so-called λ-design conjecture ([12,15]). Almost thirty years old, this famous conjecture is still open.

In this section, we are going to consider the 1-weak version of the nonuniform Fisher inequality. We are able to prove a tight bound for this problem. Somewhat surprisingly, it turns out that all optimal constructions can be characterized by Hadamard's matrices. This, at the same time, results in a new extremal set theoretic characterization of Hadamard's matrices.

Theorem 4.1. *Suppose that $L=\{\lambda\}$. If $n>2$ then $m_1(n,L)\le 2(n-1)$. Moreover, except the case $n=3$, the equality holds if and only if $n=4\lambda$ and an Hadamard's matrix of order n exists.*

It is simple to show that $m_1(1,L)=2$ and $m_1(2,L)=3$, so the bound does not hold in these cases.

In this section, $rank(M)$ will denote the rank of a matrix M. In order to prove the theorem, first we need the following lemma.

Lemma 4.2. *Let M be an n by m matrix of rank r. Denote by $M+u$ the matrix obtained from M by adding to every column a non-zero vector u. Then, $rank(M+u)\ge rank(M)-1$. The equality $rank(M+u)=rank(M)-1$ holds if and only if for any choice of r independent columns of M, the vector $-u$ and every column of M can be seen as an affine combination of the chosen columns.*

Proof. The inequality $rank(M+u)\ge r-1$ is trivial by the subadditivity of the ranks of matrices. Assume that $rank(M+u)=r-1$. Choose r independent columns $v_1,v_2,\dots,v_r$ of M arbitrarily. The n by r matrix with columns v_i+u $(1\le i\le r)$ has rank $r-1$; so, we can suppose that the first $r-1$ vectors v_i+u $(1\le i\le r-1)$ are independent and that the last vector v_r+u can be represented as a linear combination of them. So, one can find $r-1$ coefficients $\alpha_1,\alpha_2,\dots,\alpha_{r-1}$ satisfying that: $v_r+u=\sum_{i=1}^{r-1}\alpha_i(v_i+u)$, that is, $-u(\sum_{i=1}^{r-1}\alpha_i-1)=-v_r+\sum_{i=1}^{r-1}\alpha_i v_i$. Divide both sides of the last equality by $(\sum_{i=1}^{r-1}\alpha_i-1)$, we obtain $-u$ as an affine combination of the vectors v_i $(1\le i\le r)$.

Consider an arbitrary column vector v_l of M. Since the rank of M is $r-1$, the vector v_l+u can be written as a linear combination of the independent vectors

v_i+u $(1\le i\le r-1)$: $v_l+u=\sum_{i=1}^{r-1}\gamma_i(v_i+u)$. Thus we have $v_l=\sum_{i=1}^{r-1}\gamma_i v_i+(\sum_{i=1}^{r-1}\gamma_i-1)u$. Replace $u=\sum_{i=1}^{r}\beta_i v_i$, we can express v_l as a linear combination of the $v_i, 1\le i\le r-1$. The sum of the coefficients in this combination is $\sum_{i=1}^{r-1}\gamma_i+(\sum_{i=1}^{r-1}\gamma_i-1)(\sum_{i=1}^{r}\beta_i)=1$, since $\sum_{i=1}^{r}\beta_i=-1$. This completes one direction of the "if and only if" statement. The other direction is left to the reader. ∎

Remark. Let λ be a non-negative number. Consider an n by n matrix A where all diagonal entries are larger than λ and all off-diagonal entries are equal to λ. The finishing step of all algebraic proofs of the nonuniform Fisher inequality is to show that such A matrix must have full rank. The text-book approach is either to compute the determinant of A directly, or to show that A is positive definite (see [3], for instance). Lemma 4.2 offers a new proof. Let u be a vector with every coordinate equals to λ. It is obvious that $rank(A-u)=n$, since only the diagonal entries are not zero. We need only show that $rank(A)\ne rank(A-u)-1$. Since $A-u$ is diagonal with positive entries, no affine combination of its columns would give the vector $-u$, which has only negative entries. Hence, A has full rank by the criterion provided in the previous lemma.

Proof of Theorem 4.1. It is easy to show that $m_1(3,L)=4$. So from here we can assume $n>3$ and $m>4$. The case $\lambda=0$ is also very easy to analyze, so in this proof we assume $\lambda>0$. We call a member A_i *normal* if all of its intersections have λ elements. If $|A_p\cap A_q|\ne\lambda$, we call the pair (A_p,A_q) *irregular*. Apparently, a 1-weak family $\mathcal{A}$ can be partitioned into irregular pairs and a set of normal members

$$\mathcal{A}=\{A_1,A_{1'}\}\cup\{A_2,A_{2'}\}\cup\ldots\cup\{A_k,A_{k'}\}\cup\{A_{k+1},A_{k+2},\ldots,A_{k+l}\},$$

where $(A_i,A_{i'})$ $(1\le i\le k)$ are the irregular pairs and A_{k+j} $(1\le j\le l)$ are the normal members.

It is clear that if there is a member with less than λ elements, then all of its intersections are irregular; in this case, the system can have at most 2 members. So we can assume that each member has at least λ elements. The following claim deals with the case when there is a member with exactly λ elements.

Claim 1. *If there is a member with exactly λ elements, then $m<2n-2$.*

Proof. Assume that A_p has exactly λ elements. Since any normal member A_{k+j} of $\mathcal{A}$ intersects A_p in λ elements, it follows that A_{k+j} should contain A_p. Consider an irregular pair $(A_i,A_{i'})$, where $p\ne i$ and $p\ne i'$; it is trivial that both A_i and $A_{i'}$ should contain A_p by a similar reason. Using the 1-weak restricted intersection property, it is simple to check that the sets $A_{k+j}\backslash A_p$ $(j=1,\ldots,l)$ and $(A_i\cup A_{i'})\backslash A_p$ $(p\ne i, p\ne i')$ should be pairwise disjoint.

Observe that if $p\ne i$ and $p\ne i'$, then $(A_i\cup A_{i'})\backslash A_p$ should have at least two elements, since both A_i and $A_{i'}$ contains A_p and they are not identical.

Suppose that A_p is normal, i.e., $p=k+j_0$, for some j_0. The set $X\backslash A_p$ of cardinality $n-\lambda$ contains all the sets $A_{k+j}\backslash A_p$ and $(A_i\cup A_{i'})\backslash A_p$, which are pairwise disjoint. Since only $A_{k+j_0}\backslash A_p$ is empty and $|(A_i\cup A_{i'})\backslash A_p|\geq 2$ for all i, it follows that $n-\lambda\geq 2k+l-1=m-1$. Consequently, $m\leq n+1-\lambda<2(n-1)$ since $n>3$. The case that A_p is in some irregular pair could be treated similarly. This ends the proof of the claim.

By the previous claim, from here we can suppose that every set in the family has more than λ elements. Let $v_i, v_{i'}$ and u_j be the characteristic vectors of $A_i, A_{i'}$ and A_{k+j}, respectively. By the intersection property of the family we obtain the following

Claim 2. $v_iv_{i'}\neq\lambda$ *for* $1\leq i\leq k$;
$v_iv_j=v_iu_h=u_hu_g=\lambda$ *for all* i, j, h, g *where* $j\neq i'$.

Let M be the n by m matrix with columns $v_1, v_{1'}, \ldots, v_k, v_{k'}, u_1, \ldots, u_l$. Denote by J_t the all-one t by t matrix and let 1_t denote a column vector of J_t. Furthermore, let $M_1=M^TM$ and $M_2=M_1-\lambda J_m$. By Claim 2, it is clear that M_2 is the direct sum of k 2 by 2 and l 1 by 1 matrices

$$M_2 = V_1\oplus V_2\oplus\ldots\oplus V_k\oplus U_1\oplus U_2\oplus\ldots\oplus U_l,$$

where the 2 by 2 matrices correspond to the irregular pairs, and the 1 by 1 matrices correspond to the normal members.

Note that all of the U_j are non-zero since $|A_{k+j}|>\lambda$, so we have that $rank(M_2)=\sum rank(V_i)+l$. On the other hand, $n\geq rank(M_1)\geq rank(M_2)-1$. This implies that if $rank(M_2)\geq\frac{m+5}{2}$, then $m\leq 2n-3$. So, in the rest of the proof we may assume that $rank(M_2)\leq\frac{m+4}{2}$. Since $\frac{m+4}{2}<m$ and M_2 is an m by m matrix, one of the matrices V_i must have rank one. The following claim tells us when a matrix V_i (which corresponds to the irregular pair $(A_i, A_{i'})$) has rank one.

Claim 3. *V_i has rank one if and only if* $|A_i\cap A_{i'}|-\lambda=-(|A_i|-\lambda)=-(|A_{i'}|-\lambda)$.

Proof. By the definition of M_2, the entries of V_i are $|A_i|-\lambda, |A_i\cap A_{i'}|-\lambda, |A_i\cap A_{i'}|-\lambda$, and $|A_{i'}|-\lambda$ (in order from left to right and top to bottom). Claim 3 follows by a simple computation.

Observe that if the condition of Claim 3 holds for some V_i, then the vector 1_m is not contained in the subspace spanned by the column vectors of M_2. Therefore, by Lemma 4.2, $rank(M_1)=rank(M_2)$. Consequently, we have that $n\geq rank(M_2)$. So if $rank(M_2)>\frac{m+2}{2}$ again we obtain that $m<2(n-1)$. Notice that $rank(M_2)$ is at least $m/2$; thus, there remains three cases to analyze: $rank(M_2)=(m+2)/2$, $rank(M_2)=(m+1)/2$, and $rank(M_2)=m/2$.

Case 1. $rank(M_2)=\frac{m+2}{2}$. We show in this case that $m<2(n-1)$. Recall that $m=2k+l$. The structure of M_2 (as the directed sum of k 2 by 2 and l 1 by 1 matrices, each of which has rank at least 1) implies that if $rank(M_2)=(m+2)/2=k+1+\frac{l}{2}$, then one of the following two cases should hold:

(a) $k=m/2$ and exactly $m/2-1=k-1$ among the 2 by 2 matrices V_i have rank one.

(b) $k=m/2-1$ and all k 2 by 2 matrices V_i $(1\le i\le k)$ have rank one.

Consider case (a) and suppose that $rank(V_i)=1$ for every $1\le i<k$ and $rank(V_k)=2$. Let $x_i=v_i-v_{i'}$ for all $1\le i<k$. By Claim 2, it is trivial that the x_i are pairwise orthogonal and hence independent. On the other hand, with the help of Claim 3, it is not complicated to verify that each x_i is orthogonal to each of the following four vectors: $v_k, v_{k'}, v_1+v_{1'}, 1_n$. So, we obtain that $\frac{m}{2}-1=\dim(x_1,x_2,\dots,x_{k-1})\le n-\dim(v_k,v_{k'},v_1+v_{1'},1_n)=n-p$. Since v_k and $v_{k'}$ are independent, p is at least two. If $p>2$, it follows from the inequality in the previous sentence that $m<2(n-1)$. To complete the proof, we show that $p=2$ cannot occur. Assume that $p=2$; it follows that 1_n and $v_1+v_{1'}$ are linear combinations of v_k and $v_{k'}$. On the other hand, v_1, $v_{1'}$, v_k and $v_{k'}$ are (0,1) vectors; thus, the only way to have both 1_n and $v_1+v_{1'}$ as linear combinations of v_k and $v_{k'}$ is the following

$$v_k+v_{k'}=1_n=v_1+v_{1'}.$$

But then we would have $v_k 1_n=v_{k'}1_n=v_k(v_1+v_{1'})=v_{k'}(v_1+v_{1'})=2\lambda$, which implies that $|A_k|=|A_{k'}|=2\lambda$. On the other hand, A_k and $A_{k'}$ are disjoint because $v_k+v_{k'}=1_n$. So A_k and $A_{k'}$ satisfy the condition of Claim 3 and therefore $r(V_k)=1$, a contradiction.

Now let us consider case (b). Let $x_i=v_i-v_{i'}$ for all $1\le i\le k$. In this case, since $k=(m/2)-1$, we have two normal members with characteristic vectors u_1 and u_2, respectively. Again Claim 2 tells us that the vectors x_i $(1\le i\le k)$ are pairwise orthogonal. Moreover, all of them are orthogonal to u_1,u_2, and 1_n. Since u_1,u_2 are $(0,1)$ vectors and $u_1u_2=\lambda$, the three vectors u_1,u_2 and 1_n are independent. It follows that $k=\frac{m}{2}-1\le n-3$. This implies $m\le 2(n-2)$.

Case 2. $rank(M_2)=\frac{m+1}{2}$. This case occurs if and only if $k=\frac{m-1}{2}$ and each of the $V_i, i\le k$ has rank one. Define the vectors $x_i, i\le k$ as in the previous case. A similar reasoning shows that these vectors are independent. Moreover, each x_i is orthogonal to both u_1 and 1_n. Since u_1 and 1_n are independent, it is clear that $\frac{m-1}{2}\le n-2$, which implies $m\le 2n-3$.

Before continuing to Case 3, let us point out that the optimal bound $2n-2$ cannot be achieved in any of the previous cases. This bound can only be achieved in Case 3, and the analysis of this case will eventually lead to the connection with Hadamard's matrices.

Case 3. $rank(M_2)=\frac{m}{2}$. This case occurs if and only if $k=\frac{m}{2}$ and each V_i $(1\le i\le k)$ has rank one. Again let $x_i=v_i-v_{i'}$ for all $1\le i\le k$. By Claim 2, the vectors x_i and 1_n are pairwise orthogonal. The first consequence of this fact is that $\frac{m}{2}\le n-1$, or equivalently $m\le 2(n-1)$. This gives us the desired upper bound. To characterize the optimal constructions, note that the x_i are orthogonal to $v_j+v_{j'}$ for every $1\le j\le k$. So if $m=2(n-1)$ then for all j, $v_j+v_{j'}$ must be equal to 1_n, that is, every irregular pair must form a partition of X. In this case, each x_i is an $(1,-1)$

vector. Therefore, we may conclude that if an optimal construction exists then we could find an orthogonal basis of R^n consisting of the vector 1_n and $(n-1)$ $(1,-1)$ vectors. This is equivalent to saying that an Hadamard's matrix of order n exists.

To complete the proof, let us show that from any Hadamard's matrix $\{h_{ij}\}$ of order n, one can construct an optimal system. First we can assume that the first row of the matrix is all-one. Hence each of the remaining $n-1$ rows consists of $(n/2)$ 1's and $(n/2)$ (-1)'s. Let $X=\{1,2,\dots,n\}$, and set $A_i^+=\{j|h_{ij}=1\}$, $A_i^-=\{j|h_{ij}=-1\}$ for $i=2,3,\dots,n$. This way we obtain a family of $2n-2$ subsets of X. Moreover, $|A_i^+\cap A_i^-|=0$ and $|A_i^+\cap A_j^-|=|A_i^+\cap A_j^+|=|A_i^-\cap A_j^-|=n/4$, for all i and j. It is clear that each set has exactly one irregular intersection of size zero. ∎

5. Remarks and open problems

5.1. Remarks

Suppose there is a system with m members having the weak intersection property with a parameter s. Using the standard greedy algorithm, we can select a subsystem with at least $m/(s+1)$ members with no exceptional intersection. Thus, we obtain a trivial upper bound:

(5.1.0). $$m_s(n,L)\le(s+1)m_0(n,L)$$

Weak versions of the Frankl–Wilson Theorem

Let us first state the (uniform) Frank–Wilson Theorem, which is one of the strongest results in this topic.

Theorem 5.1.1. (Frankl–Wilson) *Let p be a prime number and L' a set of $l<p-1$ integers. Let k be an integer, $k\notin L'$. Assume $\mathcal{F}$ is a family of subsets of a set of n elements such that*

(i) $|E|=k \pmod{p}$ *for every member E of $\mathcal{F}$*

(ii) $|E\cap F|\in L' \pmod{p}$ *for every E and F in $\mathcal{F}$, $E\neq F$.*

Then $|\mathcal{F}|\le\binom{n}{l}$.

Let $L=\{x|x \pmod{p}\in L'\}$. In our terminology, Theorem 5.1.1 states that $m_0(n,L)\le\binom{n}{l}$, under the condition (i). Theorem 5.1.1 can be seen as a generalization of Theorem 1.3. In fact, Theorem 1.1 could also be derived from Theorem 5.1.1 as one sets $p>n$ and $l=1$. Consider now the s-weak version of the original problem which we obtain by keeping the condition (i) and replacing the condition (ii) by the following weakened condition:

(ii') For each member E, there are at most s members F so that $|E\cap F|\notin L' \pmod{p}$.

By (5.1.0), we have:

$$m_s(n,L) \le (s+1)m_0(n,L) \le (s+1)\binom{n}{l}.$$

The following construction suggests that this bound may be approximately tight.

Construction 5.1.2. Consider the following partition of the ground set X (of n elements): $X = A_0 \cup_{i=1}^{\lceil \log(s+1) \rceil} B_i$, where $|B_i| = p$ for every i and $|A_0| = n - p\lceil \log(s+1) \rceil$. Define an optimal system $\mathcal{A}_0$ with restricted intersections on A_0; $|\mathcal{A}_0| = m_0(n - p\lceil \log(s+1)\rceil, L)$ (by the definition of $m_0(n,L)$). We can now select $s+1$ different subsets of cardinality divisible by p, each is the union of some B_i. Call this system $\mathcal{B}$. Define $\mathcal{A} = \{A | A = A_0 \cup B, A_0 \in \mathcal{A}_0, B \in \mathcal{B}\}$. The family $\mathcal{A}$ has $(s+1)m_0(n-p\lceil \log(s+1)\rceil, L)$ members, and each member has s non-restricted intersections. This yields

$$(s+1)m_0(n - p\lceil \log(s+1)\rceil, L) \le m_s(n,L).$$

Together with the upper bound described in (5.1.0) we have

Fact 5.1.3.

$$(s+1)m_0(n - p\lceil \log(s+1)\rceil, L) \le m_s(n,L) \le (s+1)m_0(n,L).$$

Suppose that p, s and L are fixed and n tends to infinity; if we can show

$$\lim_{n\to\infty} m_0(n - p\lceil \log(s+1)\rceil, L)/m_0(n,L) = 1,$$

then $m_s(n,L)$ is asymptotically $(s+1)m_0(n,L)$. In fact, since s and p are fixed, it suffices to show that $\lim_{n\to\infty} m_0(n-1,L)/m_0(n,L) = 1$. Though true it seems, we do not see how to prove this.

Weak versions of the Erdős–de Bruijn Theorem

When $L = \{1\}$, we deal with the weak versions of the Erdős–de Bruijn theorem [6]. For $s = 1$ we can show that $m_1(n,L) = n+1$. Consider the ground set $X = \{0,1,\ldots,n-1\}$. The only optimal system (up to isomorphism) consists of the subsets $\{0\},\{0,1\},\ldots,\{0,n-1\},\{1,2,\ldots,n-1\}$. In the case $s=2$, the tight upper bound is roughly $1.5n$. This bound can be achieved within an additive constant by considering the system

$$\{0,1\},\{0,2\},\ldots\{0,n-1\},\{0,1,2\},\{0,3,4\},\ldots,\{0,2\lfloor (n-1)/2\rfloor -1, 2\lfloor (n-1)/2\rfloor\}.$$

The proofs of these results are rather technical and somewhat similar to that of Theorem 4.1, so we omit them here.

Weak versions of the (nonuniform) Fisher inequality

Consider the s-weak version of the Fisher inequality. By (5.1.0), it follows that $m_s(n,L) \le (s+1)n$, since $m_0(L,n) \le n$. Except the case $s=1$ discussed in Section 4, we have not found any construction which gives asymptotically $(s+1)n$. The best construction we obtain gives a family with $sn/4$ members and is described below.

Construction 5.1.4. Consider two separate projective planes P_1 and P_2 of rank p. Each P_i has $q=p^2+p+1$ points. Let the ground set X be the union of P_1 and P_2; thus, $|X|=n=2q$. Denote by $l_1^i, l_2^i, \dots l_q^i$ the lines of P_i. Set $A_{i,k} = l_i^1 \cup l_{i+k}^2$ for all $1 \le i \le q, 0 \le k \le t-1$, where $l_{q+k}^2 = l_k^2$. The system $\{A_{i,k}\}$ has qt members. If $A_{i,k}$ and $A_{i',k'}$ contain the same line then they intersect in $p+2$ points, otherwise they intersect in 2 points. So each member in the system has exactly $2t$ exceptional intersections of size $p+2$.

Twin versions of results in Section 2 and 3

Each problem considered in Section 2 and 3 has its "twin" version obtained by flipping the parity. To handle the "twin" problems, we should modify the original proofs presented here (in a fairly natural way) and use some additional (but simple) arguments. For more detail, we refer to [14], where the "twin" version of Theorem 3.1.2 is discussed.

5.2. Open questions

The theorems proven in this paper form only the "tip of an iceberg", i.e., a small part of a theory to be developed. It definitely requires more insight and more powerful tools to deal with general questions concerning systems with weakly restricted intersections. In the following we collect some questions that might shed more light on the subject.

The problem we think the most challenging is to find a sharp bound for the weak version of the Ray-Chaudhury–Wilson theorem. Let us first state this famous theorem.

Theorem . . (Ray-Chaudhury–Wilson) *Assume that L is a set of l non-negative integers, then*

$$m_0(n,L) \le \sum_{i=0}^{l} \binom{n}{i}.$$

Question 1. *Determine $m_s(n,L)$, for L being a set of l non-negative integers.*

It seems already difficult to even estimate the ratio $m_s(n,L)/m_0(n,L)$, as s and L are fixed and n tends to infinity. The bound described in (5.1.0) implies that this ratio is upper bounded by $s+1$. Theorem 4.1. asserts that if $L=\{n/4\}$ then $m_1(n,L)/m_0(n,L) \approx 2$ for infinitely many n. However, this result might be misleading. As a matter of fact, we conjecture that if $|L|$ is large compared to s, and n tends to infinity, then the ratio in question tends to 1.

Question 2. Theorems 2.1 and 2.3 give tight bounds for the weak odd town problem, when $s < 2^{n/4}$ and $s \gg 2^{n/2}$, respectively. It remains an open question to determine the tight upper-bound for the case when s is between $2^{n/4}$ and $2^{n/2}$. We think the bound $s(n-2\log s)$ is still valid for most values of s in this interval.

Question 3. We do not know if the bound in Lemma 3.2.2 is sharp, except the case $k=2$. In [14], a construction with cardinality $(k-1)(n-2\log(k-1))$ is shown. It is also interesting to find the sharp bound in Lemma 3.2.4. Such a bound would generalize Theorem 2.1.

Question 4. Consider Fact 5.1.3. To prove that $m_s(n,L)$ is approximately $(s+1)m_0(n,L)$, it suffices to show the following:

$$\lim_{n\to\infty} m_0(n-1,L)/m_0(n,L) = 1.$$

Let us mention here that the original Frankl–Wilson bound is sharp for certain pairs of set L and number k. In these cases, the limit is really 1. However, there is a chance that the Frankl–Wilson bound is not asymptotically tight for some L and k. It would be interesting to prove or disprove this.

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